Critical Points, Local Maxima, and Local Minima **(Section 4.1)** **Date:** ______________

An Old Question (to get us started). What does the derivative of a function tell us about the function and the graph of the function?

Introductory Example: On your calculator, use a "Standard Window" (ZOOM, 6:ZStandard) and graph the function:

$$f(x) = x^3 - 9x^2 - 48x + 52 .$$

If we are interested in knowing on what intervals the function is increasing and on what intervals it is decreasing, obviously this graph does not help!

So, we could "play" with the window of our calculator to get a better picture of the function, or we can solve the problem analytically!

First, we need to find $f'(x)$. $f'(x) =$ ______________________________

Second, before we find where $f'(x) > 0$ or $f'(x) < 0$, let's find where ______________. (These numbers are called **critical numbers,** or **critical points**, of the function.)

So, the critical numbers of $f(x)$ are $x =$ __________ and $x =$ __________.

Now, how can we find the intervals where the function is increasing/decreasing?

1. Let's make a chart like a number line. Label the information we just found.

2. Pick a number on the number line between each critical number and substitute it into f'. Remember, we only care if this value is positive or negative. Label this information on the chart.

Let's take this problem one step further. Can we tell where the maximum and minimum points on the graph are? How?

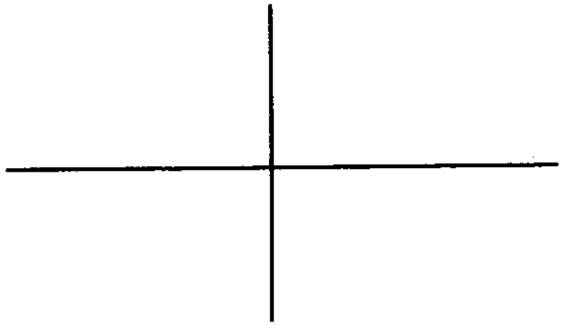
3. From the chart (number line) above, find the maximum and minimum points on the graph of the function. (These are called **local maxima** and **local minima** points of f.)

   Maxima: ____________________ Minima: __________________

Now we see why not much showed up in our first calculator graph!

4. What's the y-intercept of the graph? ____________

5. Use all of the above information, and sletch a graph of the function. Label the points that we found.



6. Adjust the plotting window on your calculator and sketch the graph again to verify our work.

Practice Problem: Given $f(x) = x^4 - 4x^3$, find the critical points, intervals where the function increases and decreases, and local maxima and minima points.

Critical points: _______________ Increasing: ________________ Decreasing: ____________________

Local Maxima: _______________ Local Minima: ________________

Let's summarize what we have learned from the examples:

1. **Definition**: A point (p , f(p)) is a **critical point** of a function f if ______________________________

2. **Property** (called the **First Derivative Test)**: Suppose p is a critical number of a function f.

   - If f changes from decreasing to increasing at p, then f has a ______________________________

   - If f changes from increasing to decreasing at p, then f has a ______________________________

The <u>second derivative</u> can also tell us whether or not there is a local maximum or minimum at a critical number p.

Recall what the second derivative tells us about a function f. Where f''> 0, f is ______________________

and where f''< 0, f is ______________________.

3. **Property** (called the **Second Derivative Test**): Suppose p is a critical number of a function f, and f'(p) = 0.

   - If f is concave up at p, then f has a ______________________ at x = p.

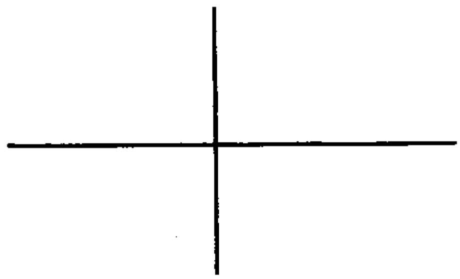
   - If f is concave down at p, then f has a ______________________ at x = p.

Example: Use the **Second Derivative Test** to verify that $f(x) = x^3 - 9x^2 - 48x + 52$ has a local maximum at x = -2 and a local minimum at x = 8, that we found in an earlier example.

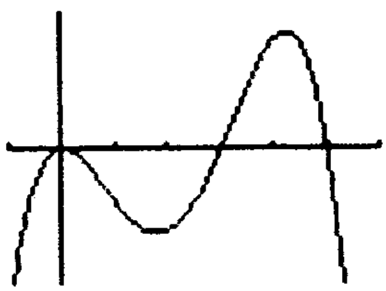
Note: Although it is a true statement that "If a function f has a local maximum or a local minimum at a number p, then p is critical number of f", it is <u>not</u> true that " If a function f has a critical number at a point p, then f is a local maximum or local minimum of f". Can you think of a simple function that verifies this?

Examples:

1. Graph a function with 3 critical points, one a local maximum, one a local minimum, and the other neither a local maximum nor a local minimum.



2. Below is a graph of f', the derivative of a function f. What are the critical points of the function? Over what intervals is the function f increasing and decreasing? For what values of x does f have a local maximum and a local minimum?



Critical numbers: ____________________

Increasing: ____________________

Decreasing: ____________________

Local Maximum: ____________________

Local Minimum: ____________________

Concavity and Inflection Points (Section 4.2) **Date:** ______________

Last class period we saw that a **critical point** was a point on the graph where the slope (f') changes sign. Now we will study the points where the concavity (f'') changes, either from concave up to concave down, or concave down to concave up.

Definition: A point at which the graph of a function f changes concavity is called an **inflection point**.

How do we locate an inflection point? Since the concavity of the graph of f changes at an inflection point, the sign of f'' changes there.

Therefore, at an inflection point, ____

Examples:

1. Find inflection points for the following functions.

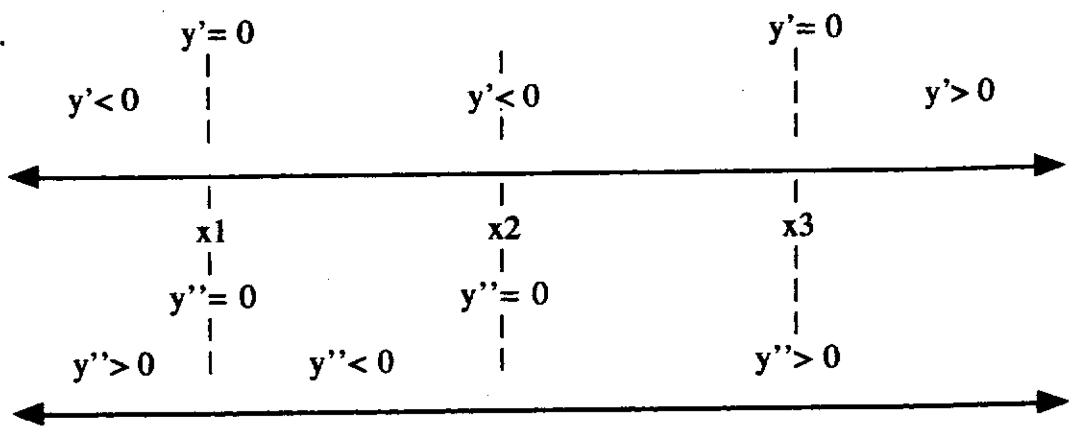
a. $f(x) = x^4 - 4x^3$

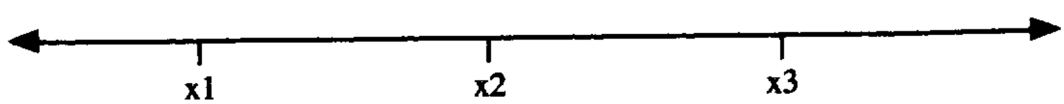
b. $g(x) = \ln(1 + x^2)$

2. Find the critical points and inflection points of $f(x) = x \cdot e^{-x}$. (This is a function we will study closely in the near future for the Pharmacy majors.)

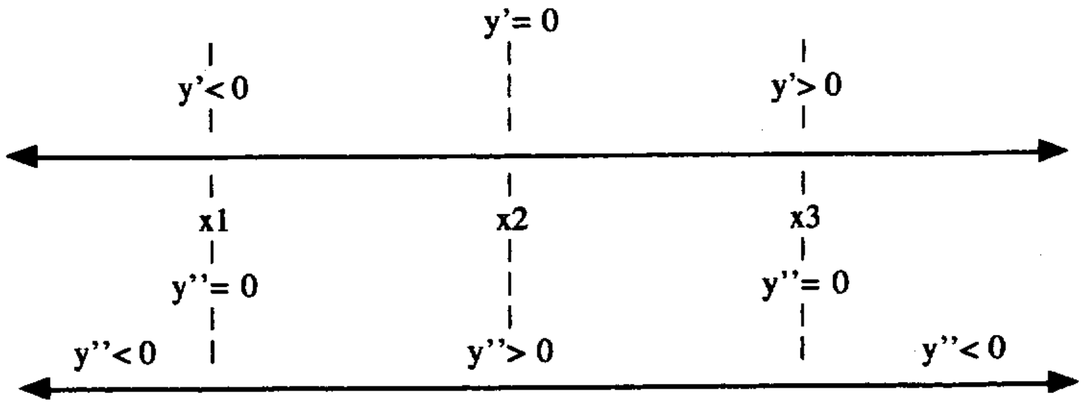
3. Using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$, sketch a possible graph of $y = f(x)$. Assume that the function is defined and continuous for all x.

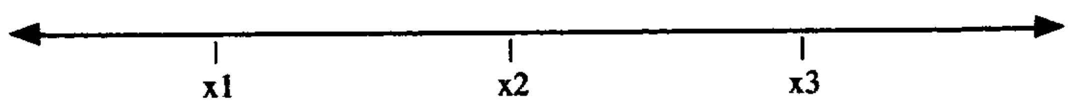
a.





b. Practice Problem:





Graphing Functions - Putting It All Together **(Section 4.2 cont.)** **Date:** ____________

Below you are given a function f, its first derivative f', and its second derivative f''.

1. Find all **intercepts** of the function.
2. Find the **critical points** of the function.
3 State intervals where the function is **increasing** and **decreasing**.
4. Find all **local maximum** and **local minimum** points.
5. Find the **possible inflection points** of the function.
6. State intervals where the function is **concave up** and **concave down**.
7. Find all **inflection** points.
8. Using the above information, **sketch the graph** of f. Label the intercepts, maximum, minimum, and inflection points.
9. You may want to check your answer by graphing the function on your calculator.

Example: $f(x) = x(x-4)^3$, $f'(x) = 4(x-1)(x-4)^2$, $f''(x) = 12(x-2)(x-4)$

Practice Problem: $f(x) = x(2x+9)^2$, $f'(x) = 3(2x+3)(2x+9)$, $f''(x) = 24(x+3)$

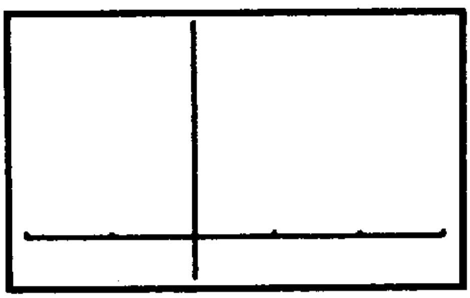
Global Maxima and Minima **(Section 4.3)** **Date:** ____________

Let's begin with an example.

Example: 1. Find all **local minima** and **local maxima** for the function $f(x) = x^4 - 2x^2 + 3$.

Local minimum: ______________ Local maximum: ____________________

2. On your calculator, sketch the graph of f on the interval [-2 , 3]. Show the graph below. Label the local maximum and local minimum.



3. We are often interested in where a function is larger or smaller than all other points on an interval. These points are called **global** (or absolute) **maxima** and **minima**. On the interval [-2 , 3], what are the **global minimum** and **global maximum** values of f?

Global minimum: ______________ Global maximum: ____________________

So, to summarize: **How do we find global maxima and global minima on a closed interval**?
(Note: A closed interval is an interval that includes the endpoints.)

Note: Given a continuous function defined on a closed interval [a , b], the function must have a global maximum and global minimum somewhere in [a , b]. (This called the **Extreme Value Theorem.)**

But what if we are interested in finding global maxima and global minima on an interval that is not a closed interval.; i.e., an interval that does not contain an endpoint. Or an interval that includes ∞.
Let's discuss this through an example.

Example: Consider the function $f(x)=\frac{\ln x}{x}$. The derivative is $f'(x)=\frac{1-\ln x}{x^2}$. (You should verify this!)

1. With your calculator, sketch f.

2. Find the critical number(s).

The critical number(s) is x = __________, and the critical point is (________ , ________)

3. State the **global maxima** and **global minima** (if they exist) on the interval indicated.

a. [1 , ∞] Max: ________ Min: ________

b. (0 , e] Max: ________ Min: ________

c. [e , ∞] Max: ________ Min: ________

d. (0 , ∞] Max: ________ Min: ________

Examples:

Graph a function with the given properties.

a. Has a local maximum at $x = -2$, a local minimum at $x = 3$, but no global maximum or minimum.

b. Has no local or global maxima or minima.

c. Has a local maximum and global maximum at $x = -2$ but no local or global minimum.

d. Has a local and global maximum at $x=-1$, and a local and global minimum at $x=3$.

Modeling and Optimization Problems **(Section 4.3 cont.)** **Date: _________**

From the previous few sections, we see that it is important to be able to find a local max/min or a global max/min of a function. But, when solving problems in the real world, it is very seldom that we are given the specific function to work with. Therefore, we need to build the function to which we will then apply our calculus concepts. These examples and problems are mainly "geometric" in nature, rather than problems from your area of study.

Examples:

1. Twenty-five feet of fence is to be put around a garden. The plans have one edge of the garden to be along the side of a house, with the fence enclosing the other three sides of a rectangle. Assume that the width of the garden is the side perpendicular to the house, and the length of the garden is the side parallel to the house,

a. Draw a figure that illustrates the problem.

b. If the width of the garden is 3 feet, what is the length of the garden? What is the area?

Width = _________ Length = _________ Area = __________

c. If the width of the garden is 10 feet, what is the length of the garden? What is the area?

Width = _________ Length = _________ Area = __________

d. If the width of the garden is x feet, write a function A(x) which represents the area of the garden as a function of the width.

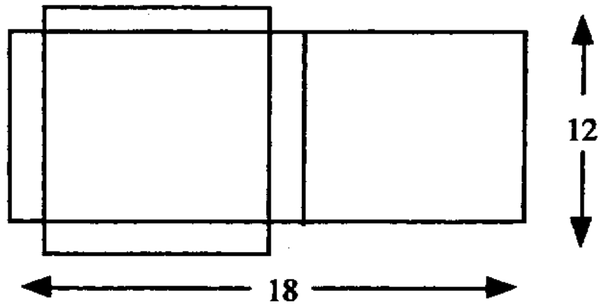
A(x) = ____

e. Analytically, find the exact width of the garden that will maximize the area. What is the length? What is the maximum area?

Width: ___________ Length: ___________ Area: ___________

f. Check your answer graphically

2. A box with a top is to be constructed out of a 12 inch by 18 inch piece of cardboard by cutting congruent squares from two of the four corners and rectangles from the other two corners, then folding up the sides (as shown below).



a. If the square in the upper left (and lower left) hand corners is **2 inches**, what is the length of the box? ____________ What is the width of the box? ______________ What is the height of the box? ______________ What is the volume of the box? __________________

b. If the square in the upper left (and lower left) hand corners is **x inches**, write an expression for the length of the box? ______________________ Write an expression for the width of the box? __________________ Write an expression for the height of the box? ___________

Write a function V(x) for the volume of the box?

V(x) = ____

c. Analytically, determine how large the cut-out squares should be to maximize the volume of the box. What is the maximum volume? Check your answer graphically.

3. You are planning to make an open rectangular box with a square base that will hold a volume of 50 cubic feet.

a. Sketch a picture of the problem.

b. If a side of the base is 2 ft, what is the height of the box? ________________ What is the surface area of the box? ____

c. If a side of the base is 5 ft, what is the height of the box? ________________ What is the surface area of the box? ____

d. If the side of the base is x ft, find an algebraic representation S(x) for the surface area of the box.

S(x) = ____

e. Using the function S(x), analytically determine the dimensions of the box if the surface area is to be as small as possible; that is, minimize the amount of material being used. What is the minimum amount of material used?

Minimum width: __________ Minimum length: ____________

Minimum height: _________ Minimum surface area: ___________

Problems for Assignment: (Do on another sheet of paper!)

4. A farmer with 5,000 feet of fencing wants to enclose a rectangular field and then divide it into two plots by adding a fence in the middle parallel to one of the sides.

a. Sketch a picture of the problem.

b. What is the area of the field if the width (one of the sides parallel to the added fence) is 500 ft?

c. What is the area of the field if the width is 1200 ft?

d. If the width of the field is x ft, find an algebraic representation A(x) for the area of the field.

e. Using the function A(x), analytically find the dimensions of the field that would maximize its area.

5. A rectangular storage container with an open top is to have a volume of 32 ft^3. The length of the base is equal to the width. Material for the base costs \$6 per square foot and material for the sides cost \$4 per square foot.

a. Sketch a picture of the problem.

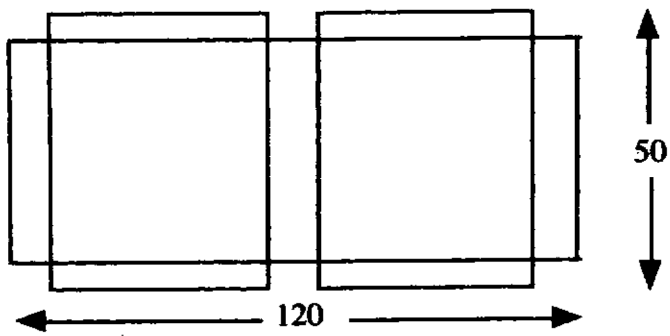
b. If the width (and length) of the base is 2 feet, what is the height of the container? What is the area of the base? What is the cost of the material to make the base? What is the area of one side? What is the cost of the material for the 4 sides? What is the cost of the material for the entire container?

c. If the width (and length) of the base is 3 feet, what is the height of the container? What is the area of the base? What is the cost of the material to make the base? What is the area of one side? What is the cost of the material for the 4 sides? What is the cost of the material for the entire container?

d. If the width (and length) of the base is x feet, write an algebraic representation for the height of the container. The area of the base. The cost of the material to make the base. The area of one side of the container. The cost of the material for the 4 sides. Finally, write a function C(x), that determines the total cost of the material for the entire container.

e. Using the function C(x), analytically find the dimensions of the container, and the cost of the materials for the cheapest such container.

6. A pizza box is to be constructed out of a 50 by 120 cm piece of cardboard by cutting six equal-sized squares in the positions shown, then folding up the sides (as shown below).



a. If the squares cut out of the six places of the cardboard measure **7 cm**, what is the length of the pizza box? What is the width of the pizza box? What is the height of the pizza box? What is the volume of the pizza box?

b. If the squares cut out of the six places of the cardboard measure **20 cm**, what is the length of the pizza box? What is the width of the pizza box? What is the height of the pizza box? What is the volume of the pizza box?

c. If the squares cut out of the six places of the cardboard measure **x cm**,

- Write an expression for the length of the pizza box?

- Write an expression for the width of the pizza box?

- Write an expression for the height of the pizza box?

- Write a function V(x) for the volume of the pizza box?

d. Find how large the cut-out squares should be to maximize the volume of the pizza box. What is the maximum volume? What is the width of this box? What is the length of this box?

A New Function - The Logistic Growth Function (Section 4.7) Date: ______________

We will introduce this new function through an interesting example!

The STARBUCKS Problem:

The first Starbucks store opened in 1971 at Pike Place Market in Seattle. The chart below shows the growth of the company from the year 1988.

Year	1988	1990	1992	1994	1996	1998	2000	2002
Number	33	84	165	425	1015	1886	3501	5886

1. Create a scatterplot of the data, with the year representing number of years from 1988; i.e. enter 1988 as 0, 1990 as 2, etc.

2. What type of function would seem to best model the growth of Starbucks? ____________________

3. Using the model $n(t) = a \cdot b^t$, analytically find a function that fits the data. (You need to find values for "a" and "b".) On your calculator, graph n(t) to see how it fits the scatterplot.

n(t) = ____________________

4. Use **ExpReg** (Exponential Regression) to find the exponential function n(t) that best fits the data. (From the Home Screen, press STAT, right arrow to CALC, Enter, 0:ExpReg.) On your calculator, graph n(t).

n(t)= ____________________

5. Using the exponential regression function, find the projected number of Starbucks in the year 2004? (Note: You may want to turn off the scatterplot and the function found in #3 above.)

n(______) = ____________________

6. Using the exponential regression function, find when the number of Starbucks will reach 11,000. Solve this problem algebraically, then check your answer graphically.

t = ____________________

7. Remember, in addition to writing an exponential function in the form $n(t) = a \cdot b^t$, we can also write the exponential function in the form $n(t) = a \cdot e^{kt}$, but we need to find the value of k. Find the value of k and rewrite the function in this form.

k≈ ____________ and n(t) = ______________________

8. Above, we determined that, according to the exponential regression model, the projected number of Starbucks in the year 2004 would be ____________. We now know that in 2003 there were 7225 Starbucks, and in 2004 there were 8337 stores. Add these values to the scatterplot. How does the exponential growth model fit the function with these new values?

9. A better fitting model now would be a **logistic function**. Using the regression capabilities of the calculator, find a logistic function (B:Logistic) that fits the data.

n(t) = ________________________________

10. Does the logistic growth model show an "upper limit" to the number of Starbucks stores? Change the window to see if the graph appears to have a horizontal aymptote as x gets bigger. If it does, what is it?

y = ______________

Where does this number appear in the logistic function? ______________________________________

11. Use the logistic function to estimate when the number of Starbucks reaches 11,000. Solve the equation graphically.

t = __________

12. Find n'(10), and interpret its meaning with units. How does this answer compare to the information from the chart on the preceding page?

The Logistic Function

A logistic function, such as that used to model the growth of Starbucks, is everywhere positive ($f(x) > 0$), and everywhere increasing, ($f'(x) > 0$). Its graph is concave up at first ($f''(x) > 0$), then becomes concave down ($f''(x) < 0$), and levels off at a horizontal asymptote.

Also, a logistic function is approximately exponential for small values of t.

Logistic functions can also be used to model the growth of a population in a closed environment, the sales of a new product, and the spread of a virus or disease.

For positive constants L, C, and k, a logistic function has the form:

$$P = f(t) = \frac{L}{1 + Ce^{-kt}}$$

Notice that the general logistic function $P = f(t) = \frac{L}{1 + Ce^{-kt}}$ has three parameters: L, C, and k.

Let's investigate the effect that two of these parameters (L and k) have on the graph of a logistic function.

Example: Consider the logistic function $P = \frac{L}{1 + 100e^{-kt}}$.

a. Let k = 1. Graph P with L = 1, L = 2, and L = 4, to see the effect of L on the function.

The value of L determines the ______________________________

Note: This value is called the **carrying capacity.**

b. Now let L = 1. Graph P with k = 1, k = 2, k = 3, to see the effect of k on the function.

The value of k determines the ______________________________

It is often important (but difficult) to predict the carrying capacity of a logistic model. For example, a manufacturing company may want to estimate the maximum potential sales of a new product.

One way of estimating the carrying capacity is to find the inflection point. At the inflection point, where the concavity changes from concave up to concave down, the slope of the curve is the largest. (For this reason, the inflection point in an economics application is called the point of diminishing returns.)

It can be shown, of course using calculus, but not easily, that the inflection point occurs where the y-value $P = \frac{L}{2}$.

Note: For the logistic function that models the growth of the Starbucks stores, $n(t) = \frac{12841}{1 + 307e^{-.397t}}$, the inflection point would occur when y = ____________. What year did this occur? ____________

Example: The table below shows the total sales (in thousands) of a new CD since it was introduced.

t (in months)	0	1	2	3	4	5	6	7
P (total sales in thousands)	0.5	2	8	33	95	258	403	496

1. After how many months does the function that models the sales of the CD change concavity? ______

2. Use this value to estimate the maximum potential sales, L. ______

3. Using a logistic regression, find the best fitting logistic function to this data.

   P(t) = ______

4. What maximum potential sales does the regression equation predict? ______

5. According to the regression equation, at what time does the point of diminishing returns occur? (Note: Watch how this is found!)

Another New Function - The Surge Function and Drug Concentration (Section 4.8) **Date:** ______________

Again, we will introduce this new function through an example.

The concentration of bemetizide (a diuretic) in the blood after a single oral dose of 25 mg is given below.

t (hours)	0	1	2	3	4	6	8	12	16
concentration (ng/ml)	0	22	57	80	82	78	60	36	25

Create a scatterplot of the data.

Functions with behavior such as this are called **surge functions.** They have equations in the form:

$$y = ate^{-bt}, \text{ where a and b are positive constants.}$$

Let's look at the "family of curves" in this form, and see what effect the parameters "a" and "b" have on the shape of the graph.

Let a = 1. Graph y with b = 1, b = 2, and b = 3, to see the effect of b on the function.

Notice that the general shape of the curve does not change as b changes, but as b increases, the curve rises for a shorter period of time and has a smaller maximum value.

Where does the maximum occur for each of the three values of b.

When b = 1, t = __________ , and when b = 2, t = ___________ , and when b = 3, t = ___________

It appears that the maximum value of $y = te^{-bt}$ ocurs at t = ___________. Let's prove this!

And, if we want to find the maximum value of the function, we can substitute 1/b for t in $y = te^{-bt}$ and evaluate y.

Conclusion: The function $y = te^{-bt}$ has a maximum at the point (________ , __________)

Now, let's look at the effect that the parameter "a" has on the shape of the graph in the function $y = ate^{-bt}$

Let b = 1. Graph y with a = 1, a = 2, and a = 3, to see the effect of a on the function.

Does it effect the x-value of the maximum point? __________ (We could prove this!)

Does it effect the y-value of the maximum? ___________ The y-value is now y = ____________.

Conclusion: The function $y = ate^{-bt}$ has a maximum at the point (_________ , _________)

This result can allow us find values of "a" and "b" in the surge function $y = ate^{-bt}$, if we know where the maximum point is. And, this would allow us to find an approximate model for the data that we have. Let's see how this works!

Since we know the maximum occurs at t = 1/b, we can solve this equation for b. b = ________

And the y-value of the maximum point occurs when y = _________. Solve this equation for a. a = _________

Let's see how this works for our "drug concentration" example. Here is the table of values again.

t (hours)	0	1	2	3	4	6	8	12	16
concentration (ng/ml)	0	22	57	80	82	78	60	36	25

The maximum value of ____________ occurs when t = _________. Use this to solve for a and b, and write the surge function that (hopefully) approximates the data.

b = ____________ , a = __________________________ , y = ___________________________

Graph it to see how it fits!

Minimum Effective Concentration: The minimum effective concentration of a drug is the blood concentration necessary to achieve a pharmacological response. The time at which this concentration is reached is referred to as **onset**, and the time that the drug concentration falls below this level, is when **termination** occurs.

Back to the example! Suppose the minimum effective concentration of bemetizide in a patient is 40 ng/ml. Find the time until the onset of effectiveness, the termination of effectiveness, and the total time that the drug is effective.

Onset: t = ___________ Termination: t = ___________ Total time of effectiveness = ___________